MLCC 2018
Regularization Network II: Kernels

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About this class

- Extend our model to deal with non linear problems
- Formulate the Representer Theorem
- Introduce kernel functions (+ examples)

Linear model...

- Data set $S = \{(x_1, y_1), \dots, (x_n, y_n)\}$ with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$
- $\hat{X} = (x_1, \dots, x_n)^\top \in \mathbb{R}^{n \times d}$ and $\hat{y} = (y_1, \dots, y_n)^\top$.
- Linear model $w \in \mathbb{R}^d$: $y = w^\top x$
- $$\min_{w \in \mathbb{R}^d} \ell(y_i, f_w(x_i)) + \lambda \|w\|^2$$

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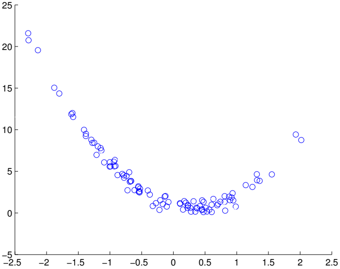
Example $d = 1$ and S as in the plot.



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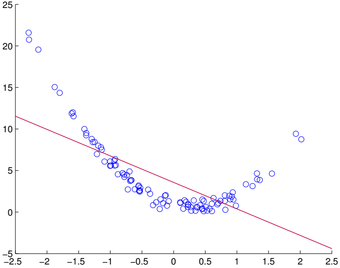

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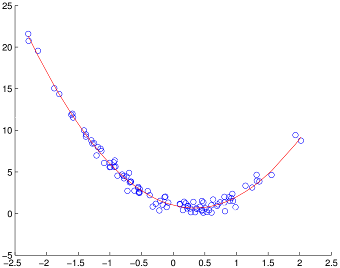

with $\phi(x) = (x^2, x, 1)^\top$ and $w = (w_1, w_2, w_3)$

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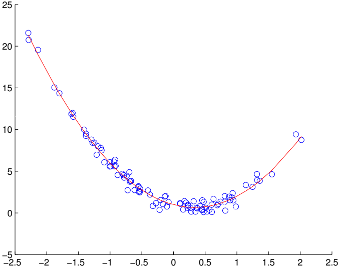

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Non linear models

- Let define $\varphi_j(x) : \mathbb{R}^d \to \mathbb{R}$ with $j \in \{1, \dots, D\}$ (in general with $D >> d$)
- $\phi : \mathbb{R}^d \to \mathbb{R}^D$ is named *feature map* with $\phi(x) = (\varphi_1(x), \dots, \varphi_D(x))^\top$.
- $w \in \mathbb{R}^D$.

Generalized linear model

$$y = w^\top \phi(x) = \sum_{j=1}^{D} w_j \varphi_j(x)$$

How to compute a non linear model (least squares)

Let define $\hat{\Phi} = (\phi(x_1), \dots, \phi(x_n))^\top \in \mathbb{R}^D$.
$\hat{\Phi}$ in generalized linear models has the same role of $\hat{X}$ in the linear models

$$w = (\hat{\Phi}^\top \hat{\Phi} + \lambda n I)^{-1} \hat{\Phi}^\top \hat{y}$$

Can we do better?
(from a computational point of view)

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Representer Theorem (in the least squares context)
There exists a $c \in \mathbb{R}^n$ such that

$$w = \hat{\Phi}^\top c = \sum_{i=1}^{n} c_i \phi(x_i),$$

in particular $c = (\hat{\Phi}\hat{\Phi}^\top + \lambda n I)^{-1} \hat{y}$.
Note that $\hat{\Phi}\hat{\Phi}^\top \in \mathbb{R}^{n \times n}$.

Sketch of the Proof

- Let $\hat{\Phi} = U\Sigma V^\top$ be the Singular Value Decomposition of $\hat{\Phi}$
- $U^\top U = I_{n\times n}$, $V^\top V = I_{n\times n}$
- $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. (Note that $\Sigma = \Sigma^\top$)

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$$w = \hat{\Phi}^\top c$$

with $c = (\hat{\Phi}\hat{\Phi}^\top + \lambda n I^\top)^{-1}\hat{y}$

Representer Theorem for general Loss Functions

For a given loss function $\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, let the problem be

$$w^* = \arg\min_{w \in \mathbb{R}^D} \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, \phi(x_i)^\top w) + \lambda \|w\|^2$$

The solution can always be written as $w^* = \hat{\Phi}^\top c$ for some coefficients vector $c = (c_1, \ldots, c_n)$

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$$w = \hat{w} \; + \; w_\perp \quad \text{for each } w \in \mathbb{R}^D$$

with $\hat{w} \in \hat{W}$ and $v^\top w_\perp = 0$ for each $v \in \hat{W}$.

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Therefore for any x_i with $i \in \{1, \dots, n\}$

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Therefore the problem become

$$w^* = \arg\min_{w \in \mathbb{R}^D} \frac{1}{n} \sum_{i=1}^n V(y_i, \phi(x_i)^\top \hat{w}) + \lambda \|w\|^2$$

Moreover, considering that $\hat{w}^\top w_\perp = 0$ we have

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$$w^* = \hat{\Phi}^\top c$$

for some $c \in \mathbb{R}^n$.

Why we need Kernels...

Let analyze the RLS solution for the Generalized Linear model, we have

$$f(x) = \phi(x)^\top \hat{\Phi}^\top (\hat{\Phi}\hat{\Phi}^\top + \lambda n I)^{-1} \hat{y}$$

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$f(x)$ is expressed only by using inner products between feature vectors

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In this way we have

$$f(x) = \hat{K}_x^\top (\hat{K} + \lambda n I)^{-1} \hat{y}$$

with $\hat{K}_x = (K(x, x_1), \ldots, K(x, x_n)), \quad (\hat{K})_{ij} = K(x_i, x_j)$.

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The same holds for general loss functions indeed

$$f(x) = \phi(x)^\top w^* = \phi(x)^\top \hat{\Phi}^\top c = \hat{K}_x^\top c = \sum_{i=1}^n c_i K(x, x_i).$$

Examples of Kernel: Linear Kernel

For $x, z \in \mathbb{R}^d$

$$K(x, z) = x^\top z$$

Proof

$$K(x, z) = \phi(x)^\top \phi(z)$$

with $\phi : \mathbb{R}^d \to \mathbb{R}^d$ defined as

$$\phi(x) = x.$$

Examples of Kernel: Affine Kernel

For $x, z \in \mathbb{R}^d$

$$K(x, z) = x^\top z + \alpha^2$$

Proof

$$K(x, z) = \phi(x)^\top \phi(z)$$

with $\phi : \mathbb{R}^d \to \mathbb{R}^{d+1}$ defined as

$$\phi(x) = (x, \alpha).$$

Examples of Kernel: Polynomial Kernel of degree p

For $p \in \mathbb{N}$

$$K(x,z) = (xz+1)^p \quad \text{with } x,z \in \mathbb{R}$$

Proof

$$(xz+1)^p = \sum_{k=0}^{p} q_{p,k}(xz)^k = \phi(x)^\top \phi(z)$$

with $q_{p,k} = \frac{p!}{k!(p-k)!}$ and $\phi : \mathbb{R} \to \mathbb{R}^{p+1}$ defined as

$$\phi(x) = (\sqrt{q_{p,0}}, \sqrt{q_{p,1}}x, \sqrt{q_{p,2}}x^2, \ldots, \sqrt{q_{p,k}}x^k, \ldots, \sqrt{q_{p,p}}x^p)$$

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For $x, z \in \mathbb{R}^d$ it is defined as

$$K(x,z) = (x^\top z + 1)^p$$

Examples of Kernel: Polynomial Kernel of any degree

For $x, z \in [0, 1]$ and $0 < \alpha < 1$

$$K(x, z) = \frac{1}{1 - \alpha^2 xz}$$

Proof

$$\frac{1}{1 - \alpha xz} = \sum_{k=0}^{\infty} (\alpha^2 xz)^k = \phi(x)^\top \phi(z)$$

with $\phi : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ defined as

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Kernel - Characterization

$K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a *Kernel* if it behaves like an inner product that is

1. it is symmetric

$$K(x, z) = K(z, x) \quad \text{for all } x, z \in \mathbb{R}^d$$

2. it is positive definite (p.d.).

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2. it is positive definite (p.d.). For any $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}^d$ define $\hat{K}$ as $(\hat{K})_{ij} = K(x_i, x_j)$.

Kernel - Characterization

$K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a *Kernel* if it behaves like an inner product that is

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$$K \text{ is p.d.} \quad \text{iff} \quad \hat{K} \text{ is p.d. for any } n \in \mathbb{N}, x_1, \dots, x_n \in \mathbb{R}^d$$

The first is easy to check, the second is quite difficult!

Kernel properties

Let $K_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, K_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, K_3 : \mathbb{R}^t \times \mathbb{R}^t$ be Kernels and $x, x' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^t$ and $\alpha, \beta > 0$ then the following are Kernels too

1. $\alpha K_1(x, x') + \beta K_2(x, x')$
2. $K_1(x, x')K_2(x, x')$
3. $p(K_1(x, x'))$ for any p a function whose polynomial expansion has only non-negative coefficients
4. $f(x)K_1(x, x')f(x')$ for any $f : \mathbb{R}^d \to \mathbb{R}$
5. $\frac{K_1(x,x')}{\sqrt{K_1(x,x)K_1(x',x')}}$
6. $K_3(\psi(x), \psi(x))$ for any $\psi : \mathbb{R}^d \to \mathbb{R}^t$
7. $\alpha K_1(x, x') + \beta K_3(z, z')$
8. $K_1(x, x')K_3(z, z')$

Gaussian Kernel

Let $x, x' \in \mathbb{R}^d$ and $\sigma > 0$, the gaussian kernel is

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But $K_3 = K$ indeed

$$K_3(x,x') = f(x)e^{\frac{x^\top x'}{\sigma^2}}f(x') = e^{-\frac{x^\top x + x'^\top x' - 2x^\top x'}{2\sigma^2}} = e^{\frac{-\|x-x'\|^2}{2\sigma^2}} = K(x,x')$$

Wrapping up

In this class we discussed how to deal with high dimensional non linear problems (feature maps and kernels). We also introduced the Represented Theorem.

Next class

Beyond prediction, we will focus more on data exploration and learning of interpretable models.