

# Introductory Machine Learning Notes<sup>1</sup>

Lorenzo Rosasco

`lrosasco@mit.edu`

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These are the notes for the course master level course *Intelligent Systems and Machine Learning (ISML)- Module II* for the Computer Science Degree the University of Genova. They are my attempt to extract essential machine learning notes for beginners. They are still a draft and will often be updated. Likely they won't be typos free for a while. They are dry and they lack examples to complement and illustrate the general ideas. Notably, the notes also lack references, that will hopefully be added soon. In fact, large portion of the material presented draws from various sources on the web, perhaps somewhat integrated in an original way. Other times they follow closely some source. Some pics are wildly downloaded from the web. Finally, I would like to acknowledge that the mathematical appendix is due to Andre Wibisono's notes for the math camp of the 9.520 course at MIT.

ABSTRACT. Machine Learning has become a key to develop intelligent systems and analyze data in science and engineering. Machine learning engines enable systems such as Siri, Kinect or Google self driving car, to name a few examples. At the same time machine learning methods help helping deciphering the information in our dna and make sense of the flood of information gathered on the web. This notes provide an introduction to basic powerful methods that form the core of modern machine learning.

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## CHAPTER 1

# Statistical Learning Theory

Machine Learning deals with systems that are trained from data rather than being explicitly programmed. Here we describe the data model considered in statistical learning theory.

### 1.1. Data

The data, called *training set*, is a set of  $n$  input-output pairs,

$$S = \{(x_1, y_1), \dots, (x_n, y_n)\}.$$

Each pair is called an example. We consider the approach to machine learning based on the so called *learning from examples* paradigm.

Given the training set, the goal is to *learn* a corresponding input-output relation. To make sense of this task we have to postulate the existence of a model for the data. The model should take into account the possible *uncertainty* in the task and in the data.

### 1.2. Probabilistic Data Model

The inputs belong to an input space  $X$ , we assume throughout that  $X \subseteq \mathbb{R}^D$ . The outputs belong to an output space  $Y$ . We consider several possible situations: regression  $Y \subseteq \mathbb{R}$ , binary classification  $Y = \{-1, 1\}$  and multi-category (multiclass) classification  $Y = \{1, 2, \dots, T\}$ . The space  $X \times Y$  is called the *data space*.

We assume there exists a fixed unknown data distribution  $p(x, y)$  according to which the data are identically and independently distributed (i.i.d.). The probability distribution  $p$  models different sources of uncertainty. We assume that it factorizes as  $p(x, y) = p_X(x)p(y|x)$ , where

- the conditional distribution  $p(y|x)$ , see Figure 1, can be seen as a form of *noise* in the output. For example, in regression the following model is often considered  $y = f^*(x) + \epsilon$ , where  $f^*$  is a fixed unknown function and  $\epsilon$  is random noise, e.g. standard Gaussian  $\mathcal{N}(0, \sigma I)$ ,  $\sigma \in [0, \infty)$ . In classification, a noiseless situation corresponds to  $p(1|x) = 1$  or 0 for all  $x$ .
- The marginal distribution  $p_X(x)$  models uncertainty in the sampling of the input points.

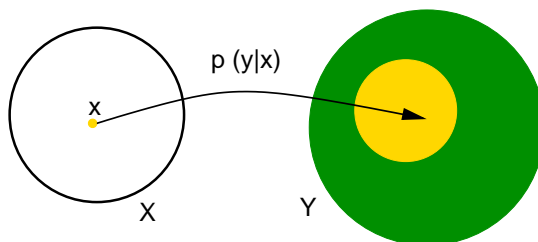


FIGURE 1. For each input  $x$  there is a distribution of possible outputs  $p(y|x)$ .

### 1.3. Loss Function and Expected Risk

The goal of learning is to estimate the “best” input-output relation— rather than the whole distribution  $p$ .

More precisely, we need to fix a *loss function*

$$\ell : Y \times Y \rightarrow [0, \infty),$$

which is a (point-wise) measure of the error  $\ell(y, f(x))$  we incur in when predicting  $f(x)$  in place of  $y$ . Given a loss function, the “best” input-output relation is the *target function*  $f^* : X \rightarrow Y$  minimizing the *expected loss (or expected risk)*

$$\mathcal{E}(f) = \mathbb{E}[\ell(y, f(x))] = \int dx dy p(x, y) \ell(y, f(x)).$$

which can be seen as a measure of the error on past as well as future data. The target function cannot be computed since the probability distribution  $p$  is unknown. A (good) learning algorithm should provide a solution that behaves similarly to the target function, and predict/classify well new data. In this case, we say that the algorithm *generalizes*.

### 1.4. Stability, Overfitting and Regularization

A *learning algorithm* is a procedure that given a training set  $S$  computes an estimator  $f_S$ . Ideally an estimator should *mimic* the target function, in the sense that  $\mathcal{E}(f_S) \approx \mathcal{E}(f^*)$ . The latter requirement needs some care since  $f_S$  depends on the training set and hence is random. For example one possibility is to require an algorithm to be good in *expectation*, in the sense that

$$\mathbb{E}_S \mathcal{E}(f_S) - \mathcal{E}(f^*),$$

is small.

More intuitively, a good learning algorithm should be able to describe well (fit) the data, and at the same time be stable with respect to noise and sampling. Indeed, a key to ensure good generalization property is to avoid overfitting, that is having estimators which are highly dependent on the data (unstable), possibly with a low error on the training set and yet a large error on future data. Most learning algorithms depend one (or more) regularization parameter that control the trade-off between data-fitting and stability. We broadly refer to this class of approaches as regularization algorithms, their study is our main topic of discussion.

## Local Methods

We describe a simple yet efficient class of algorithms, the so called memory based learning algorithms, based on the principle that nearby input points should have the similar/same output.

### 2.1. Nearest Neighbor

Consider a training set

$$S = \{(x_1, y_1), \dots, (x_n, y_n)\}.$$

Given an input  $\bar{x}$ , let

$$i' = \arg \min_{i=1, \dots, n} \|\bar{x} - x_i\|^2$$

and define the nearest neighbor (NN) estimator as

$$\hat{f}(\bar{x}) = y_{i'}.$$

Every new input point is assigned the same output as its nearest input in the training set. We add few comments. First, while in the above definition we simply considered the Euclidean norm, the method can be promptly generalized to consider other measure of similarity among inputs. For example if the input are binary strings, i.e.  $X = \{0, 1\}^D$ , one could consider the Hamming distance

$$d_H(x, \bar{x}) = \frac{1}{D} \sum_{j=1}^D \mathbf{1}_{[x^j \neq \bar{x}^j]}$$

where  $x^j$  is the  $j$ -th component of a string  $x \in X$ .

Second, the complexity of the algorithm for predicting any new point is  $O(nD)$ —recall that the complexity of multiplying two  $D$ -dimensional vectors is  $O(D)$ . Finally, we note that NN can be fairly sensitive to noise.

### 2.2. K-Nearest Neighbor

Let  $d_{\bar{x}} = (\|\bar{x} - x_i\|^2)_{i=1}^n$  be the array of distances of a new point  $\bar{x}$  to the input points in the training set. Let  $s_{\bar{x}}$  be the above array sorted in increasing order and  $I_{\bar{x}}$  the corresponding vector of indices. Let  $K_{\bar{x}} = \{I_{\bar{x}}^1, \dots, I_{\bar{x}}^K\}$  be the array of the first  $K$  entries of  $I_{\bar{x}}$ . The  $K$ -nearest neighbor estimator (KNN) is defined as,

$$\hat{f}(\bar{x}) = \sum_{i' \in K_{\bar{x}}} y_{i'},$$

or  $\hat{f}(\bar{x}) = \frac{1}{K} \sum_{i' \in K_{\bar{x}}} y_{i'}$ . In classification KNN can be seen as a *voting scheme* among the  $K$  nearest neighbors and  $K$  is taken to be odd to avoid ties. The parameter  $K$  controls the stability of the KNN estimate: when  $K$  is small the algorithm is sensitive to the data (and simply reduces to NN for  $K = 1$ ). When  $K$  increases the estimator becomes more stable. In classification, it eventually simply becomes the ratio of the number of elements for each class. The question of how to best choose  $K$  will be the subject of a future discussion.

### 2.3. Parzen Windows

In KNN each of the  $K$  neighbors has equal weights in determining the output of a new point. A more general approach is to consider estimators of the form,

$$\hat{f}(\bar{x}) = \frac{\sum_{i=1}^n y_i k(\bar{x}, x_i)}{\sum_{i=1}^n k(\bar{x}, x_i)},$$

where  $k : X \times X \rightarrow [0, 1]$  is a suitable function, which can be seen as a similarity measure on the input points. The function  $k$  defines a *window* around each point and is sometimes called a Parzen window. A classification rule is obtained considering the sign of  $\hat{f}(\bar{x})$ .

Many examples of  $k$  depend on the distance  $\|x - x'\|$ ,  $x, x' \in X$ . For example we can consider

$$k(x', x) = \mathbf{1}_{\|x-x'\| \leq r}.$$

This choice induce a Parzen window analogous to KNN, here the parameter  $K$  is replaced by the *radius*  $r$ . More generally it is interesting to have a decaying weight for point which are further away. For example considering

$$k(x', x) = (1 - \|x - x'\|)_+ \mathbf{1}_{\|x-x'\| \leq r},$$

where  $(a)_+ = a$ , if  $a > 0$  and  $(a)_+ = 0$ , otherwise. Another possibility is to consider fast decaying functions such as a Gaussian

$$k(x', x) = e^{-\|x-x'\|^2/2\sigma^2}.$$

or an exponential

$$k(x', x) = e^{-\|x-x'\|/\sqrt{2}\sigma}.$$

In all the above methods there is a parameter  $r$  or  $\sigma$  that controls the influence that each neighbor has on the prediction.

#### 2.4. High Dimensions

The following simple reasoning highlights a phenomenon which is typical of dealing with high dimensional learning problems. Consider a unit cube in  $D$  dimensions, and a smaller cube of edge  $e$ . How shall we choose  $e$  to capture 1% of the volume of the larger cube? Clearly, we need  $e = \sqrt[D]{.01}$ . For example  $e = .63$  for  $D = 10$  and  $e = .95$  for  $D = 100$ . The edge of the *small* cube is virtually the *same* length of that of the *large* cube. The above example illustrates how in high dimensions our intuition of neighbors and neighborhoods is challenged.



## Bias Variance and Cross-Validation

Here we ask the question of how to choose  $K$ : is there an optima choice of  $K$ ? Can it be computed in practice? Towards answering these questions we investigate theoretically the question of how  $K$  affects the performance of the KNN algorithm.

### 3.1. Tuning and Bias Variance Variance Decomposition

For the sake of simplicity we consider a regression model

$$y_i = f_*(x_i) + \delta_i, \quad i = 1, \dots, n$$

where  $\mathbf{E}\delta_i = 0$  and  $\mathbf{E}\delta_i^2 = \sigma^2$ , for all  $i = 1, \dots, n$  and denote by  $S$  the corresponding training set. Moreover, we consider the least square loss function to measure errors, so that the performance of the KNN algorithm is given by the expected loss

$$\mathbf{E}_S \mathbf{E}_{x,y} (y - \hat{f}_K(x))^2 = \mathbf{E}_x \underbrace{\mathbf{E}_S \mathbf{E}_{y|x} (y - \hat{f}_K(x))^2}_{\varepsilon(K)}.$$

To get an insight on how to choose  $K$ , we analyze theoretically how this choice influences the expected loss. In fact, in the following we simplify the analysis considering the performance of KNN  $\varepsilon(K)$  at a given point  $x$ .

First, note that

$$\varepsilon(K) = \sigma^2 + \mathbf{E}_S \mathbf{E}_{y|x} (f_*(x) - \hat{f}_K(x))^2,$$

where  $\sigma^2$  can be seen as an irreducible error term. Second, to study the latter term we introduce the *expected* KNN algorithm,

$$\mathbf{E}_{y|x} \hat{f}_K(x) = \frac{1}{K} \sum_{\ell \in K_x} f_*(x_\ell).$$

We have

$$\mathbf{E}_S \mathbf{E}_{y|x} (f_*(x) - \hat{f}_K(x))^2 = \underbrace{(f_*(x) - \mathbf{E}_S \mathbf{E}_{y|x} \hat{f}_K(x))^2}_{Bias} + \underbrace{\mathbf{E}_S \mathbf{E}_{y|x} (\mathbf{E}_{y|x} \hat{f}_K(x) - \hat{f}_K(x))^2}_{Variance}$$

Finally, we have

$$\varepsilon(K) = \sigma^2 + (f_*(x) - \frac{1}{K} \sum_{\ell \in K_x} f_*(x_\ell))^2 + \frac{\sigma^2}{K}$$

### 3.2. The Bias Variance Trade-Off

We are ready to discuss the behavior of the (point-wise) expected loss of the KNN algorithm as a function of  $K$ . As it is clear from the above equation, the variance decreases with  $K$ . The bias is likely to increase with  $K$ , if the function  $f_*$  is suitably smooth. Indeed, for small  $K$  the few closest neighbors to  $x$  will have values close to  $f_*(x)$ , so their average will be close to  $f_*(x)$ . Whereas, as  $K$  increases neighbors will be further away and their average might move away from  $f_*(x)$ . A larger bias variance is preferred when data are few/noisy to achieve a better control of the variance, whereas the bias can be decreased as more data become available, hence reducing the variance. For any given training set, the

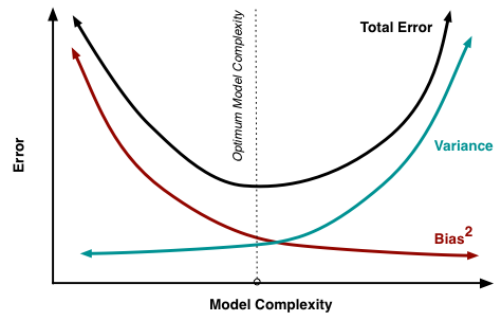


FIGURE 1. The Bias-Variance Tradeoff. In the KNN algorithm the parameter  $K$  controls the achieved (model) complexity.

best choice for  $K$  would be the one striking the optimal trade-off between bias and variance (that is the value minimizing their sum).

### 3.3. Cross Validation

While instructive, the above analysis is not directly useful in practice since the data distribution, hence the expected loss, is not accessible. In practice, data driven procedures are used to find a proxy for the expected loss. The simplest such procedure is called hold-out cross validation. Part of the training  $S$  set is hold-out, to compute a (holdout) error to be used as a proxy of the expected error. An empirical bias variance trade-off is achieved choosing the value of  $K$  that achieves minimum hold-out error. When data are scarce the hold-out procedure, based on a simple "two ways split" of the training set, might be unstable. In this case, so called  $V$ -fold cross validation is preferred, which is based on multiple data splitting. More precisely, the data are divided in  $V$  (non overlapping) sets. Each set is hold-out and used to compute an hold-out error which is eventually averaged to obtained the final  $V$ -fold cross validation error. The extreme case where  $V = n$  is called leave-one-out cross validation.

**3.3.1. Conclusions: Beyond KNN.** Most of the above reasonings hold for a large class of learning algorithms beyond KNN. Indeed, many (most) algorithms depend on one or more parameter controlling the bias-variance tradeoff.

## Regularized Least Squares

In this class we introduce a class of learning algorithms based Tikhonov regularization, a.k.a. penalized empirical risk minimization and regularization. In particular, we focus on the algorithm defined by the square loss.

### 4.1. Regularized Least Squares

We consider the following algorithm

$$(4.1) \quad \min_{w \in \mathbb{R}^D} \frac{1}{n} \sum_{i=1}^n (y_i - w^T x_i)^2 + \lambda w^T w, \lambda \geq 0.$$

A motivation for considering the above scheme is to view the empirical error

$$\frac{1}{n} \sum_{i=1}^n (y_i - w^T x_i)^2,$$

as a proxy for the expected error

$$\int dx dy p(x, y) (y - w^T x)^2.$$

The term  $w^T w$  is a regularizer and help preventing overfitting.

The term  $w^T w = \|w\|^2$  is called regularizer and controls the stability of the solution. The parameter  $\lambda$  balances the error term and the regularizer. Algorithm (4.1) is an instance of Tikhonov regularization, also called penalized empirical risk minimization. We have implicitly chosen the space of possible solution, called the hypotheses space, to be the space of linear functions, that is

$$H = \{f : \mathbb{R}^D \rightarrow \mathbb{R} : \exists w \in \mathbb{R}^D \text{ such that } f(x) = x^T w, \forall x \in \mathbb{R}^D\},$$

so that finding a function  $f_w$  reduces to finding a vector  $w$ . As we will see in the following, this seemingly simple example will be the basis for much more complicated solutions.

### 4.2. Computations

In this case it is convenient to introduce the  $n$  times  $D$  matrix  $X_n$ , where the rows are the input points, and the  $n$  by 1 vector  $Y_n$  where the entries are the corresponding outputs. With this notation

$$\frac{1}{n} \sum_{i=1}^n (y_i - w^T x_i)^2 = \frac{1}{n} \|Y_n - X_n w\|^2.$$

A direct computation shows that the gradient with respect to  $w$  of the empirical risk and the regularizer are respectively

$$-\frac{2}{n} X_n^T (Y_n - X_n w), \quad \text{and,} \quad 2w.$$

Then, setting the gradient to zero, we have that the solution of regularized least squares solves the linear system

$$(X_n^T X_n + \lambda n I) w = X_n^T Y_n.$$

Several comments are in order. First, several methods can be used to solve the above linear systems, Choleski decomposition being the method of choice, since the matrix  $X_n^T X_n + \lambda I$  is symmetric and positive definite. The complexity of the method is essentially  $O(nd^2)$  for training and  $O(d)$  for testing. The parameter  $\lambda$  controls the *invertibility* of the matrix  $(X_n^T X_n + \lambda n I)$ .

### 4.3. Interlude: Linear Systems

Consider the problem

$$Ma = b,$$

where  $M$  is a  $D$  by  $D$  matrix and  $a, b$  vectors in  $\mathbb{R}^D$ . We are interested in determining  $a$  satisfying the above equation given  $M, b$ . If  $M$  is invertible, the solution to the problem is

$$a = M^{-1}b.$$

- If  $M$  is a diagonal  $M = \text{diag}(\sigma_1, \dots, \sigma_D)$  where  $\sigma_i \in (0, \infty)$  for all  $i = 1, \dots, D$ , then

$$M^{-1} = \text{diag}(1/\sigma_1, \dots, 1/\sigma_D), \quad (M + \lambda I)^{-1} = \text{diag}(1/(\sigma_1 + \lambda), \dots, 1/(\sigma_D + \lambda))$$

- If  $M$  is symmetric and positive definite, then considering the eigendecomposition

$$M^{-1} = V\Sigma V^T, \quad \Sigma = \text{diag}(\sigma_1, \dots, \sigma_D), \quad VV^T = I,$$

then

$$M^{-1} = V\Sigma^{-1}V^T, \quad \Sigma^{-1} = \text{diag}(1/\sigma_1, \dots, 1/\sigma_D),$$

and

$$(M + \lambda I)^{-1} = V\Sigma_\lambda^{-1}V^T, \quad \Sigma_\lambda^{-1} = \text{diag}(1/(\sigma_1 + \lambda), \dots, 1/(\sigma_D + \lambda))$$

The ratio  $\sigma_D/\sigma_1$  is called the condition number of  $M$ .

### 4.4. Dealing with an Offset

When considering linear models, especially in relatively low dimensional spaces, it is interesting to consider an offset, that is  $w^T x + b$ . We shall ask the question of how to estimate  $b$  from data. A simple idea is to simply augment the dimension of the input space, considering  $\tilde{x} = (x, 1)$  and  $\tilde{w} = (w, b)$ . While this is fine if we do not regularize, if we do then we still tend to prefer linear functions passing through the origin, since the regularizer becomes

$$\|\tilde{w}\|^2 = \|w\|^2 + b^2.$$

In general we might not have reasons to believe that the model should pass through the origin, hence we would like to consider an offset and still regularize considering only  $\|w\|^2$ , so that the offset is not penalized. Note that the regularized problem becomes

$$\min_{(w,b) \in \mathbb{R}^{D+1}} \frac{1}{n} \sum_{i=1}^n (y_i - w^T x_i - b)^2 + \lambda \|w\|^2.$$

The solution of the above problem is particularly simple when considering least squares. Indeed, in this case it can be easily proved that a solution  $w^*, b^*$  of the above problem is given by

$$b^* = \bar{y} - \bar{x}^T w^*$$

where  $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$ ,  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$  and  $w^*$  solves

$$\min_{w \in \mathbb{R}^{D+1}} \frac{1}{n} \sum_{i=1}^n (y_i^c - w^T x_i^c)^2 + \lambda \|w\|^2.$$

where  $y_i^c = y_i - \bar{y}$  and  $x_i^c = x_i - \bar{x}$  for all  $i = 1, \dots, n$ .

## Regularized Least Squares Classification

In this class we introduce a class of learning algorithms based Tikhonov regularization, a.k.a. penalized empirical risk minimization and regularization. In particular, we focus on the algorithm defined by the square loss.

While least squares are often associated to regression problem, we next discuss their interpretation in the context of binary classification and discuss an extension to multi class classification.

### 5.1. Nearest Centroid Classifier

Let's consider a classification problem and assume that there is an equal number of point for class 1 and  $-1$ . Recall that the nearest centroid rule is given by

$$\text{sign}h(x), \quad h(x) = \|x - m_{-1}\|^2 - \|x - m_1\|^2$$

where

$$m_1 = \frac{2}{n} \sum_{i | y_i=1} x_i, \quad m_{-1} = \frac{2}{n} \sum_{i | y_i=-1} x_i.$$

It is easy to see that we can write,

$$h(x) = x^T w + b, \quad w = m_1 - m_{-1}, \quad b = -(m_1 - m_{-1})^T m,$$

where

$$m = m_1 + m_{-1} = \frac{1}{n} \sum_{i=1}^n x_i.$$

In a compact notation we can write,

$$h(x) = (x - m)^T (m_1 - m_{-1}).$$

### 5.2. RLS for Binary Classification

If we consider an offset, the classification rule given by RLS is

$$\text{sign}f(x), \quad f(x) = x^T w + b,$$

where

$$b = -m^T w,$$

since  $\frac{1}{n} \sum_{i=1}^n y_i = 0$  by assumption, and

$$w = (\bar{X}_n^T \bar{X}_n + \lambda n I)^{-1} \bar{X}_n^T Y_n = \left( \frac{1}{n} \bar{X}_n^T \bar{X}_n + \lambda I \right)^{-1} \frac{1}{n} \bar{X}_n^T Y_n,$$

with  $\bar{X}_n$  the *centered* data matrix having rows  $x_i - m, i = 1, \dots, n$ .

It is easy to show a connection between the RLS classification rule and the nearest centroid rule. Note that,

$$\frac{1}{n} \bar{X}_n^T Y_n = \frac{1}{n} X_n^T Y_n = m_1 - m_{-1},$$

so that, if we let  $C_\lambda = \frac{1}{n} \bar{X}_n^T \bar{X}_n + \lambda I$

$$b = -m^T C_\lambda^{-1} (m_1 - m_{-1})$$

and

$$f(x) = (x - m)^T C_\lambda^{-1} (m_1 - m_{-1})$$

If  $\lambda$  is large then  $(\frac{1}{n} X_n^T X_n + \lambda I) \approx \lambda I$ , and we see that

$$f(x) \sim \frac{1}{\lambda} h(x) \Leftrightarrow \text{sign}f(x) = \text{sign}h(x).$$

If  $\lambda$  is small  $C_\lambda \sim C = \frac{1}{n} \overline{X}_n^T \overline{X}_n$ , the inner product  $x^T w$  is replaced with a new inner product  $(x - m)^T C^{-1} (x - m)$ . The latter is the so called Mahalanobis distance. If we consider the eigendecomposition of  $C = V \Sigma V^T$  we can better understand the effect of the new inner product. We have

$$f(x) = (x - m)^T V \Sigma^{-1} \lambda^{-1} V^T (m_1 - m_{-1}) = (\tilde{x} - \tilde{m})^T (\tilde{m}_1 - \tilde{m}_{-1}),$$

where  $\tilde{u} = \Sigma^{1/2} V^T u$ . The data are rotated and then stretched in directions where the eigenvalues are small.

### 5.3. RLS for Multiclass Classification

RLS can be adapted to problem with  $T > 2$  classes considering

$$(5.1) \quad (X_n^T X_n + \lambda n I) W = X_n^T Y_n.$$

where  $W$  is a  $D$  by  $T$  matrix, and  $Y_n$  is a  $n$  by  $T$  matrix where the  $i$ -th column has entry 1 if the corresponding input belongs to the  $i$ -th class and  $-1$  otherwise. If we let  $w_t, t = 1, \dots, T$ , denote the columns of  $W$  then the corresponding classification rule  $c : X \rightarrow \{1, \dots, T\}$  is

$$c(x) = \arg \max_{t=1, \dots, T} x^T W^t$$

The above scheme can be seen as a reduction scheme from multi class to a collection of binary classification problems. Indeed, the solution of 5.1 can be shown to solve the minimization problem

$$\min_{W^1, \dots, W^T} \sum_{t=1}^T \left( \frac{1}{n} \sum_{i=1}^n (y_i^t - x_i^T W^t)^2 + \lambda \|W^t\|^2 \right).$$

where  $y_i^t = 1$  if the  $x_i$  belong to class  $t$  and  $y_i^t = -1$ , otherwise. The above minimization can be done separately for all  $w_i, i = 1, \dots, T$ . Each minimization problem can be interpreted as performing a "one vs all" binary classification.

## CHAPTER 6

### Feature, Kernels and Representer Theorem

In this class we introduce the concepts of feature map and kernel that allow to generalize Regularization Networks, and not only, well beyond linear models. Our starting point will be again Tikhonov regularization,

$$(6.1) \quad \min_{w \in \mathbb{R}^D} \frac{1}{n} \sum_{i=1}^n \ell(y_i, f_w(x_i)) + \lambda \|w\|^2.$$

#### 6.1. Feature Maps

A feature map is a map

$$\Phi : X \rightarrow F$$

from the input space into a new space called feature space where there is a scalar product  $\Phi(x)^T \Phi(x')$ . The feature space can be infinite dimensional and the following notation is used for the scalar product  $\langle \Phi(x), \Phi(x') \rangle_F$ .

**6.1.1. Beyond Linear Models.** The simplest case is when  $F = \mathbb{R}^p$ , and we can view the entries  $\Phi(x)^j$ ,  $j = 1, \dots, p$  as novel measurements on the input points. For illustrative purposes consider  $X = \mathbb{R}^2$ . An example of feature map could be  $x = (x_1, x_2) \mapsto \Phi(x) = (x_1^2, \sqrt{2}x_1x_2, x_2^2)$ . With this choice if we now consider

$$f_w(x) = w^T \Phi(x) = \sum_{j=1}^p w^j \Phi(x)^j$$

we effectively have that the function is no longer linear but it is a polynomial of degree 2. Clearly the same reasoning holds for much more general choice of measurements (features), in fact *any* finite set of measurements. Although seemingly simple, the above observation allows to consider very general models. Figure 1 gives a geometric interpretation of the potential effect of considering a feature map. Points which are not easily classified by a linear model, can be easily classified by a *linear model in the feature space*. Indeed, the model is no longer linear in the original input space.

**6.1.2. Computations.** While feature maps allow to consider nonlinear models, the computations are essentially the same as in the linear case. Indeed, it is easy to see that the computations considered for linear models, under different loss functions, remain unchanged, as long as we change  $x \in \mathbb{R}^D$  into  $\Phi(x) \in \mathbb{R}^p$ . For example, for least squares we simply need to replace the  $n$  by  $D$  matrix  $X_n$  with a new  $n$  by  $p$  matrix  $\Phi_n$ , where each row is the image of an input point in the feature space as defined by the feature map.

#### 6.2. Representer Theorem

In this section we discuss how the above reasoning can be further generalized. The key result is that the solution of regularization problems of the form (6.1), can always be written as

$$(6.2) \quad \hat{w}^T = \sum_{i=1}^n x_i^T c_i,$$

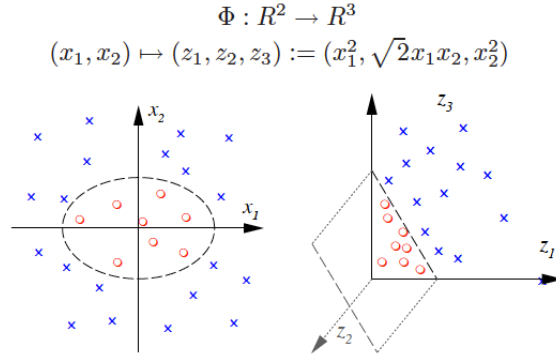


FIGURE 1. A pictorial representation of the potential effect of considering a feature map in a simple two dimensional example.

where  $x_1, \dots, x_n$  are the inputs in the training set and  $c = (c_1, \dots, c_n)$  a set of coefficients. The above result is an instance of the so called representer theorem. We first discuss this result in the context of RLS.

**6.2.1. Representer Theorem for RLS.** The result follows noting that the following equality holds,

$$(6.3) \quad (X_n^T X_n + \lambda n I)^{-1} X_n^T = X_n^T (X_n X_n^T + \lambda n I)^{-1},$$

so that we have,

$$w = X_n^T \underbrace{(X_n X_n^T + \lambda n I)^{-1} Y_n}_c = \sum_{i=1}^n x_i^T c_i.$$

Equation (6.3) follows from considering the SVD of  $X_n$ , that is  $X_n = U \Sigma V^T$ . Indeed we have  $X_n^T = V \Sigma U^T$  so that

$$(X_n^T X_n + \lambda n I)^{-1} X_n^T = V (\Sigma^2 + \lambda)^{-1} \Sigma U^T$$

and

$$X_n^T (X_n X_n^T + \lambda n I)^{-1} = V \Sigma (\Sigma^2 + \lambda)^{-1} U^T.$$

**6.2.2. Representer Theorem Implications.** Using Equation (7.2) it possible to show how the vector  $c$  of coefficients can computed considering different loss functions. In particular, for the square loss the vector of coefficients satisfies the following linear system

$$(K_n + \lambda n I)c = Y_n.$$

where  $K_n$  is the  $n$  by  $n$  matrix with entries  $(K_n)_{i,j} = x_i^T x_j$ . The matrix  $K_n$  is called the *kernel matrix* and is symmetric and positive semi-definite.

### 6.3. Kernels

One of the main advantages of using the representer theorem is that the solution of the problem depends on the input points only through inner products  $x^T x'$ . Kernel methods can be seen as replacing the inner product with a more general function  $K(x, x')$ . In this case, the representer theorem (7.2), that is  $f_w(x) = w^T = \sum_{i=1}^n x_i^T c_i$ , becomes

$$(6.4) \quad \hat{f}(x) = \sum_{i=1}^n K(x_i, x) c_i.$$

and we can promptly derive kernel versions of the Regularization Networks induced by different loss functions.

The function  $K$  is often called a kernel and to be admissible it should *behave like* an inner product. More precisely it should be: 1) symmetric, and 2) positive definite, that is the kernel matrix  $K_n$  should be positive semi-definite for any set of  $n$  input points. While the symmetry property is typically easy to check, positive semi definiteness is trickier. Popular examples of positive definite kernels include:

- the linear kernel  $K(x, x') = x^T x'$ ,
- the polynomial kernel  $K(x, x') = (x^T x' + 1)^d$ ,



- the Gaussian kernel  $K(x, x') = e^{-\frac{\|x-x'\|^2}{2\sigma^2}}$ ,

where the last two kernels have a tuning parameter, the degree and Gaussian width, respectively.

A positive definite kernel is often called a *reproducing kernel* and is a key concept in the theory of reproducing kernel Hilbert spaces.

We end noting that there are some basic operations that can be used to build new kernels. In particular it is easy to see that, if  $K_1, K_2$  are reproducing kernels then  $K_1 + K_2$  is also a kernel.



## Regularization Networks

In this class we introduce a class of learning algorithms based on Tikhonov regularization, a.k.a. penalized empirical risk minimization and regularization. In particular, we study common computational aspects of these algorithms introducing the so called representer theorem.

### 7.1. Empirical Risk Minimization

Among different approaches to design learning algorithms, empirical risk minimization (ERM) is probably the most popular one. The general idea behind this class of methods is to consider the empirical error

$$\hat{\mathcal{E}}(f) = \frac{1}{n} \sum_{i=1}^n \ell(y_i, f(x_i)),$$

as a proxy for the expected error

$$\mathcal{E}(f) = \mathbb{E}[\ell(y, f(x))] = \int dx dy p(x, y) \ell(y, f(x)).$$

Recall that  $\ell$  is a loss function and measure the price we pay predicting  $f(x)$  when in fact the right label is  $y$ . Also, recall that the expected error cannot be directly computed since the data distribution is fixed but unknown.

In practice, to turn the above idea into an actual algorithm we need to fix a suitable hypotheses space  $H$  on which we will minimize  $\hat{\mathcal{E}}$ .

### 7.2. Hypotheses Space

The hypotheses space should be such that computations are feasible, at the same time it should be *rich* since the complexity of the problem is not known a priori. As we have seen, the simplest example of hypotheses space is the space of linear functions, that is

$$H = \{f : \mathbb{R}^D \rightarrow \mathbb{R} : \exists w \in \mathbb{R}^D \text{ such that } f(x) = x^T w, \forall x \in \mathbb{R}^D\}.$$

Each function  $f$  is defined by a vector  $w$  and we let  $f_w(x) = x^T w$ . We have also seen how we can vastly extend the class of functions we can consider by introducing a feature map

$$\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^p,$$

where typically  $p \gg D$ , and considering functions of the form  $f_w(x) = \Phi(x)^T w$ . We have also seen how this model can be pushed further considering so called reproducing kernels

$$K : \mathbb{R}^D \times \mathbb{R}^D \rightarrow \mathbb{R}$$

that is symmetric and positive definite functions, implicitly defining a feature map via the equation

$$\Phi(x)^T \Phi(x') = K(x, x').$$

If the hypotheses space is rich enough, solely minimizing the empirical risk is not enough to ensure a generalizing solution. Indeed, simply solving ERM would lead to estimators which are highly dependent on the data and could overfit. Regularization is a general class of techniques that allow to restore stability and ensure generalization.

### 7.3. Tikhonov regularization and Representer Theorem

We consider the following *Tikhonov* regularization scheme,

$$(7.1) \quad \min_{w \in \mathbb{R}^D} \hat{\mathcal{E}}(f_w) + \lambda \|w\|^2.$$

The above scheme describes a large class of methods sometimes called Regularization Networks. The term  $\|w\|^2$  is called regularizer and controls the stability of the solution. The parameter  $\lambda$  balances the error term and the regularizer.

Different classes of methods are induced by the choice of different loss functions. In the following, we will see common aspects and differences in considering different loss functions.

There is no general computation scheme to solve problems of the form (7.1), and the actual solution for each algorithm depend on the considered loss function. However, we show next that for linear function the solution of problem (7.1) can always be written as

$$(7.2) \quad w = X_n^t c, \quad f(x) = \sum_{i=1}^n x_i^T x c_i$$

where  $X_n$  is the  $n$  by  $D$  data matrix and  $c = (c_1, \dots, c_n)$ . This allows on the one hand to reduce computational complexity when  $n \ll D$ , or  $n \ll p$  in the case of feature map.

**7.3.1. Representer Theorem for General Loss Functions.** Here we discuss the general proof of the representer theorem for loss function other than the square loss.

- The vectors of the form (7.2) form a linear subspace  $\widehat{W}$  of  $\mathbb{R}^D$ . Hence for every  $v \in \mathbb{R}^D$  we have the decomposition  $w = \hat{w} + \hat{w}^\perp$ , where  $\hat{w} \in \widehat{W}$  and  $\hat{w}^\perp$  belongs to the space  $\widehat{W}^\perp$  of vectors orthogonal to those in  $\widehat{W}$ , i.e.

$$(7.3) \quad \hat{w}^T \hat{w}^\perp = 0.$$

- The following is the key observation: for all  $i = 1, \dots, n$   $x_i \in \widehat{W}$ , so that

$$f_w(x_i) = x_i^T w = x_i^T (\hat{w} + \hat{w}^\perp) = x_i^T \hat{w}.$$

It follows that the empirical error depends only on  $\hat{w}$ !

- For the regularizer we have

$$\|w\|^2 = \|\hat{w} + \hat{w}^\perp\|^2 = \|\hat{w}\|^2 + \|\hat{w}^\perp\|^2,$$

because of (7.3). Clearly the above expression is minimized if we take  $\hat{w}^\perp = 0$ .

The theorem is hence proved, the first term in (7.1) depends only on vector of the form (7.2) and the same form is the best to minimize the second term

### 7.4. Loss Functions and Target Functions

It is useful to recall that different loss function might define different goal via the corresponding target functions.

A simple calculation shows what is the target function corresponding to the square loss. Recall that the target function minimize the expected squared loss error

$$\mathcal{E}(f) = \int p(x, y) dx dy (y - f(x))^2 = \int p(x) dx \int p(y|x) dy (y - f(x))^2.$$

To simplify the computation we let

$$f^*(x) = \arg \min_{a \in \mathbb{R}} \int p(y|x) dy (y - a)^2,$$

for all  $x \in X$ . It is easy to see that the solution is given by

$$f^*(x) = \int dy p(y|x) y.$$

In classification

$$f^*(x) = 2p - 1, \quad p = p(1|x),$$

which justify taking the sing of  $f$ .

Similarly we can derive the target function of the logistic loss function,

$$f^*(x) = \arg \min_{a \in \mathbb{R}} \int p(y|x) dy \log(1 + e^{-ya}) = \arg \min_{a \in \mathbb{R}} p \log(1 + e^{-a}) + (1 - p) \log(1 + e^a).$$

We can simply take the derivative and set it equal to zero,

$$p \frac{-e^{-a}}{(1 + e^{-a})} + (1 - p) \frac{e^a}{(1 + e^a)} = -p \frac{1}{(1 + e^{-a})} + (1 - p) \frac{e^a}{(1 + e^a)},$$

so that

$$p = \frac{e^a}{(1 + e^a)} \implies a = \log \frac{p}{1 - p}$$

A similar computation allows to



## Logistic Regression

We consider logistic regression, that is *Tikhonov* regularization

$$(8.1) \quad \min_{w \in \mathbb{R}^D} \hat{\mathcal{E}}(f_w) + \lambda \|w\|^2, \quad \hat{\mathcal{E}}(f_w) = \frac{1}{n} \sum_{i=1}^n \ell(y_i, f_w(x_i))$$

where the loss function is  $\ell(y, f_w(x)) = \log(1 + e^{-yf_w(x)})$ , namely the logistic loss function.

Since the logistic loss function is differentiable the natural candidate to compute a minimizer is a the gradient descent algorithm which we describe next.

### 8.1. Interlude: Gradient Descent and Stochastic Gradient

Before starting let's recall the following basic definition

- Gradient of  $G : \mathbb{R}^D \rightarrow \mathbb{R}$ ,

$$\nabla G = \left( \frac{\partial G}{\partial w^1}, \dots, \frac{\partial G}{\partial w^D} \right)$$

- Hessian of  $G : \mathbb{R}^D \rightarrow \mathbb{R}$ ,

$$H(G)_{i,j} = \frac{\partial^2 G}{\partial w^i \partial w^j}$$

- Jacobian of  $F : \mathbb{R}^D \rightarrow \mathbb{R}^D$

$$J(F)_{i,j} = \frac{\partial F^i}{\partial w^j}$$

Note that  $H(G) = J(\nabla G)$ .

Consider the minimization problem

$$\min_{w \in \mathbb{R}^D} G(w) \quad G : \mathbb{R}^D \rightarrow \mathbb{R}$$

when  $G$  is a differentiable (strictly convex) function. A general approach to find an approximate solution of the problem is the *gradient descent* (GD) algorithm, based on the following iteration

$$(8.2) \quad w_{t+1} = w_t - \gamma \nabla G(w_t)$$

for a suitable initialization  $w_0$ . Above  $\nabla G(w)$  is the gradient of  $G$  at  $w$  and  $\gamma$  is a positive constant (or a sequence) called the step-size. Choosing the step-size appropriately ensures the iteration to converge to a minimizing solution. In particular, a suitable choice can be shown to be

$$\gamma = 1/L,$$

where  $L$  is the *Lipschitz constant* of the gradient, that is  $L$  such that

$$\|\nabla G(w) - \nabla G(w')\| \leq L \|w - w'\|.$$

It can be shown that  $L$  is less or equal than the biggest eigenvalue of the Hessian  $H(G)(w)$  for all  $w$ . The term descent comes from the fact that it can be shown that

$$G(w_t) \geq G(w_{t+1}).$$

A related technique is called stochastic gradient or also incremental gradient. To describe this method, we consider an objective function is the form

$$G(w) = \sum_{i=1}^n g_i(w), \quad g_i : \mathbb{R}^D \rightarrow \mathbb{R}, \quad i = 1, \dots, n,$$

so that  $\nabla G(w) = \sum_{i=1}^n \nabla g_i(w)$ . The stochast gradient algorithm corresponds to replacing (8.2) with

$$w_{t+1} = w_t - \gamma \nabla g_{i_t}(w_t)$$

where  $i_t$  denotes a deterministic or stochastic sequence of indices. In this case, the step size needs to be chosen as sequence  $\gamma_t$  going to zero but not too fast. For example the choice  $\gamma_t = 1/t$  can be shown to suffice.

### 8.2. Regularized Logistic Regression

The corresponding regularized empirical risk minimization problem is called regularized logistic regression. Its solution can be computed via gradient descent or stochastic gradient. Note that

$$\nabla \hat{\mathcal{E}}(f_w) = \frac{1}{n} \sum_{i=1}^n x_i \frac{-y_i e^{-y_i x_i^T w_{t-1}}}{1 + e^{-y_i x_i^T w_{t-1}}} = \frac{1}{n} \sum_{i=1}^n x_i \frac{-y_i}{1 + e^{y_i x_i^T w_{t-1}}}$$

so that, for  $w_0 = 0$ , the gradient descent algorithm applied to (8.1) is

$$w_t = w_{t-1} - \gamma \left( \frac{1}{n} \sum_{i=1}^n x_i \frac{-y_i}{1 + e^{y_i x_i^T w_{t-1}}} + 2\lambda w_{t-1} \right)$$

for  $t = 1, \dots, T$ , where

$$\frac{1}{n} \sum_{i=1}^n \frac{-y_i x_i e^{-y_i x_i^T w}}{1 + e^{-y_i x_i^T w}} + 2\lambda w = \nabla(\hat{\mathcal{E}}(f_w) + \lambda \|w\|^2)$$

A direct computation shows that

$$J(\nabla \hat{\mathcal{E}}(f_w)) = \frac{1}{n} \sum_{i=1}^n x_i x_i^T \ell''(y_i w^T x_i) + 2\lambda I$$

where  $\ell''(a) = \frac{e^{-a}}{(1+e^{-a})^2} \leq 1$  is the second derivative of the function  $\ell(a) = \log(1 + e^{-a})$ . In particular it can be shown that

$$L \leq \sigma_{\max}\left(\frac{1}{n} X_n^T X_n + 2\lambda I\right)$$

where  $\sigma_{\max}(A)$  is the largest eigenvalue of a (symmetric positive semidefinite) matrix  $A$ .

### 8.3. Kernel Regularized Logistic Regression

The vector of coefficients can be computed by the following iteration

$$c_t = c_{t-1} - \gamma B(c_{t-1}), \quad t = 1, \dots, T$$

for  $c_0 = 0$ , and where  $B(c_{t-1}) \in \mathbb{R}^n$  with

$$B(c_{t-1})^i = -\frac{1}{n} \frac{y_i}{1 + e^{y_i \sum_{k=1}^n x_k^T x_i c_{t-1}^k}} + 2\lambda c_{t-1}^i.$$

Here again we choose a constant step-size. Note that

$$\sigma_{\max}\left(\frac{1}{n} X_n^T X_n + \lambda I\right) = \sigma_{\max}\left(\frac{1}{n} X_n X_n^T + \lambda I\right) = \sigma_{\max}\left(\frac{1}{n} K_n + \lambda I\right).$$

### 8.4. Logistic Regression and confidence estimation

We end recalling that a main feature of logistic regression is that, as discussed, The solution of logistic regression can be shown to have probabilistic interpretation, in fact it can be derived from the following model

$$p(1|x) = \frac{e^{x^T w}}{1 + e^{x^T w}}$$

where the right hand side is called logistic function. This latter observation can be used to deduce a confidence from the on each prediction of the logistic regression estimator.



## From Perceptron to SVM

We next introduce the support vector machine discussing one of the most classical learning algorithms, namely the perceptron algorithm.

### 9.1. Perceptron

The perceptron algorithm finds a linear classification rule according to the following iterative procedure. Set  $w_0 = 0$  and update

$$w_i = w_{i-1} + \gamma y_i w^T x_i, \text{ if } y_i w^T x_i \leq 0$$

and let  $w_i = w_{i-1}$  otherwise. In words, if an example is correctly classified, then the perceptron does not do anything. If the perceptron incorrectly classifies a training example, each of the input weights is moved a little bit in the correct direction for that training example.

The above procedure can be seen as the stochastic (sub) gradient associated to the objective function

$$\sum_{i=1}^n | -y_i w^T x_i |_+$$

where the  $|a|_+ = \max\{0, a\}$ . Indeed if  $y_i w^T x_i < 0$ , then  $| -y_i w^T x_i |_+ = -y_i w^T x_i$  and  $\nabla | -y_i w^T x_i |_+ = -y_i x_i$ , if  $y_i w^T x_i > 0$ , then  $| -y_i w^T x_i |_+ = 0$  hence  $\nabla | -y_i w^T x_i |_+ = 0$ . Clearly an off-set can also be considered, replacing  $w^T x$  by  $w^T x + b$  and analogous iteration can be derived.

The above method can be shown to converge for  $\gamma = \text{const.}$  if the data are linearly separable. If the data are not separable with a constant step size the perceptron will typically cycle. Moreover, the perceptron does not implement any specific form of regularization so in general it is prone to overfit the data.

### 9.2. Margin

The quantity  $\alpha = y w^T x$  defining the objective function of the perceptron is a natural error measure and is sometimes called the *functional margin*. Next we look at a geometric interpretation of the functional margin that will lead to a different derivation of Tikhonov regularization for the so called hinge loss function. We begin considering a binary classification problem where the classes are linearly separable.

Consider the decision surface  $D = \{x : w^T x = 0\}$  defined by a vector  $w$  and  $x$  such that  $w^T x > 0$ . It is easy to check that, the projection of  $x$  on  $D$  is a point  $x_w$  satisfying,

$$x_w = x - \beta \frac{w}{\|w\|}$$

where  $\beta$  is the distance between  $x$  and  $D$ . Clearly  $x_w \in D$  so that

$$w^T (x - \beta \frac{w}{\|w\|}) = 0 \Leftrightarrow \beta = \frac{w^T}{\|w\|} x.$$

If  $x$  is  $x$  such that  $w^T x < 0$  then  $\beta = -\frac{w^T}{\|w\|} x$ , so that generally we have

$$\beta = y \frac{w^T}{\|w\|} x$$

The above quantity is often called the geometric margin and clearly if  $\|w\| = 1$  it coincides with the geometric margin. Note that the margin is scale invariant, in the sense that  $\beta = y \frac{w^T}{\|w\|} x = y \frac{2w^T}{\|2w\|} x$ , as is the decision rule  $\text{sign}(w^T x)$ .

### 9.3. Maximizing the Margin

Maximizing the margin is a natural approach to select a linear separating rule in the separable case. More precisely consider

$$(9.1) \quad \begin{aligned} \beta_w &= \min_{i=1, \dots, n} \beta_i, \quad \beta_i = y_i \frac{w^T x_i}{\|w\|}, \quad i = 1, \dots, n, \\ \max_{w \in \mathbb{R}^D} \quad &\beta_w, \quad \text{subj. to,} \quad \beta_w \geq 0, \quad \|w\| = 1. \end{aligned}$$

Note that the last constraint is needed to avoid the solution  $w = \infty$  (check what happens if you consider a solution  $w$  and then scale it by a constant  $kw$ ).

In the following we manipulate the above expression to obtain a problem of the form

$$\min_{w \in \mathbb{R}^D} F(w), \quad Aw + c \geq 0,$$

where  $F$  is convex,  $A$  is a matrix and  $c$  a vector. These are convex programming problem which can efficiently solved.

We begin by rewriting problem (9.1) by introducing a *dummy* variable  $\beta = \beta_w$  to obtain

$$\max_{(w, \beta) \in \mathbb{R}^{D+1}} \beta, \quad \text{subj. to,} \quad y_i \frac{w^T x_i}{\|w\|} \geq \beta; \beta \geq 0, \|w\| = 1$$

(we are basically using the definition of minimum as the maximum of the infimal points). We next would like to avoid the constraint  $\|w\| = 1$ . It can be shown that the above problem is equivalent to considering

$$\max_{(w, \alpha) \in \mathbb{R}^{D+1}} \frac{\alpha}{\|w\|}, \quad \text{subj. to,} \quad y_i w^T x_i \geq \alpha; \alpha \geq 0.$$

with  $\beta = \frac{\alpha}{\|w\|}$ , where the key idea is that the latter problem is scale invariant. More precisely that we can always restrict ourselves to  $\|w\| = 1$  by appropriately rescaling the solutions. Using again scale invariance (check what happens if you consider a solution  $w$  and then scale it by a constant  $(kw, k\alpha)$ ), without loss of generality we can fix  $\alpha = 1$  to obtain

$$\max_{w \in \mathbb{R}^D} \frac{1}{\|w\|}, \quad \text{subj. to,} \quad y_i w^T x_i \geq 1, \quad i = 1, \dots, n,$$

or equivalently

$$(9.2) \quad \min_{w \in \mathbb{R}^D} \frac{1}{2} \|w\|^2, \quad \text{subj. to,} \quad y_i w^T x_i \geq 1, \quad i = 1, \dots, n,$$

In the above reasoning we assumed data to be separable if this is not the case one could considering *slack* variables  $\xi = (\xi_1, \dots, \xi_n)$  to relax the constraints in the above problem, considering

$$(9.3) \quad \min_{w \in \mathbb{R}^D, \xi \in \mathbb{R}^n} \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \xi_i, \quad \text{subj. to,} \quad y_i w^T x_i \geq 1 - \xi_i, \quad \xi_i \geq 0, \quad i = 1, \dots, n.$$

### 9.4. From Max Margin to Tikhonov Regularization

Note that  $\xi_i = \max\{0, 1 - y_i w^T x_i\} = |1 - y_i w^T x_i|_+$ , for all  $i = 1, \dots, n$ . Then if we set  $\lambda = \frac{1}{2Cn}$ , we have that problem (9.3) is equivalent to

$$\min_{w \in \mathbb{R}^D, \xi \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^n |1 - y_i w^T x_i|_+ + \lambda \|w\|^2.$$

### 9.5. Computations

The derivation of a solution to the SVM problem requires notions of convex optimization, specifically considering so called Lagrangian duality. Indeed, it can be shown that the solution of problem (9.3) is of the form

$$w = \sum_{i=1}^n y_i \alpha_i x_i$$

where the coefficients  $\alpha^i$  for  $i = 1, \dots, n$  are given by the solution of the so called dual problem,

$$(9.4) \quad \min_{\alpha \in \mathbb{R}^n} \quad \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i,j=1}^n \alpha_i \alpha_j y_i y_j x_i^T x_j, \quad \text{subject to, } 0 \leq \alpha_i \leq C, \quad i = 1, \dots, n,$$

where in particular it can be shown that

$$\alpha_i = 0 \implies y_i w^T x_i \geq 1.$$

### 9.6. Dealing with an off-set

Finally, it can be shown that the above reasoning can be generalized to consider an offset, that is  $w^T x + b$ , in which case we simply have to add the constraint

$$\sum_{i=1}^n y_i \alpha_i x_i = 0$$

to the dual problem (9.4).



## Dimensionality Reduction

In many practical applications it is of interest to reduce the dimensionality of the data. In particular, this is useful for data visualization, or for investigating the “effective” dimensionality of the data. This problem is often referred to as dimensionality reduction and can be seen as the problem of defining a map

$$M : X = \mathbb{R}^D \rightarrow \mathbb{R}^k, \quad k \ll D,$$

according to some suitable criterion.

### 10.1. PCA & Reconstruction

PCA is arguably the most popular dimensionality reduction procedure. It is a data driven procedure that given an (unsupervised) sample  $S = (x_1, \dots, x_n)$  derive a dimensionality reduction defined by a linear map  $M$ . PCA can be derived from several prospective and here we give a geometric/analytical derivation.

We begin by considering the case where  $k = 1$ . We are interested into finding the single most relevant dimension according to some suitable criterion. Recall that, if  $w \in \mathbb{R}^D$  with  $\|w\| = 1$ , then the (orthogonal) projection of a point  $x$  on  $w$  is given by  $(w^T x)w$ . Consider the problem of finding the direction  $p$  which allows the best possible average reconstruction of the training set, that is the solution of the problem

$$(10.1) \quad \min_{w \in \mathbb{S}^{D-1}} \frac{1}{n} \sum_{i=1}^n \|x_i - (w^T x_i)w\|^2,$$

where  $\mathbb{S}^{D-1} = \{w \in \mathbb{R}^D \mid \|w\| = 1\}$  is the sphere in  $D$  dimensions. The norm  $\|x_i - (w^T x_i)w\|^2$  measures how much we lose by projecting  $x$  along the direction  $w$ , and the solution  $p$  to problem (10.1) is called the first principal component of the data. A direct computation shows that  $\|x_i - (w^T x_i)w\|^2 = \|x_i\|^2 - (w^T x_i)^2$ , so that problem (10.1) is equivalent to

$$(10.2) \quad \max_{w \in \mathbb{S}^{D-1}} \frac{1}{n} \sum_{i=1}^n (w^T x_i)^2.$$

This latter observation is useful for two different reasons that we discuss in the following.

### 10.2. PCA and Maximum Variance

If the data are centered, that is  $\bar{x} = \frac{1}{n} \sum x_i = 0$ , problem (10.2) has the following interpretation: we look for the direction along which the data have (on average) maximum variance. Indeed, we can interpret the term  $(w^T x)^2$  as the variance of  $x$  in the direction  $w$ . If the data are not centered, to keep this interpretation we should replace problem (10.2) with

$$(10.3) \quad \max_{w \in \mathbb{S}^{D-1}} \frac{1}{n} \sum_{i=1}^n (w^T (x_i - \bar{x}))^2,$$

which corresponds to the original problem on the centered data  $x^c = x - \bar{x}$ . In the terms of problem (10.1) it is easy to see that this corresponds to considering

$$(10.4) \quad \min_{w, b \in \mathbb{S}^{D-1}} \frac{1}{n} \sum_{i=1}^n \|x_i - ((w^T (x_i - b))w + b)\|^2.$$

where  $((w^T (x_i - b))w + b)$  is an affine transformation (rather than an orthogonal projection).

### 10.3. PCA and Associated Eigenproblem

A simple further manipulation allows to write problem (10.2) as an eigenvalue problem. Indeed, using the symmetry of the inner product we have

$$\frac{1}{n} \sum_{i=1}^n (w^T x_i)^2 = \frac{1}{n} \sum_{i=1}^n w^T x_i w^T x_i = \frac{1}{n} \sum_{i=1}^n w^T x_i x_i^T w = w^T \left( \frac{1}{n} \sum_{i=1}^n x_i x_i^T \right) w$$

so that problem (10.2) can be written as

$$(10.5) \quad \max_{w \in \mathbb{S}^{D-1}} w^T C_n w, \quad C_n = \frac{1}{n} \sum_{i=1}^n x_i x_i^T.$$

We need two observations. First, in matrix notation  $C_n = \frac{1}{n} \sum_{i=1}^n X_n^T X_n$  and it is easy to see that  $C_n$  is symmetric and positive semi-definite. If the data are centered the matrix  $C_n$  is the so called covariance matrix. Clearly the objective function in (10.5) can be written as

$$\frac{w^T C_n w}{w^T w}$$

where the latter quantity is the so called Rayleigh quotient. Note that, if  $C_n u = \lambda u$  then  $\frac{u^T C_n u}{u^T u} = \lambda$ , since the eigenvector  $u$  normalized. In fact, it is possible to show that the Rayleigh quotient achieves its maximum at a vector which corresponds to the maximum eigenvalue of  $C_n$  (the proof of this latter fact uses basic results in linear programming). Then computing the first principal component of the data reduced to computing the biggest eigenvalue of the covariance and the corresponding eigenvector.

### 10.4. Beyond the First Principal Component

Next, we discuss how the above reasoning can be generalized to  $k > 1$ , that is more than one principle component. The idea is simply to iterate the above reasoning to describe the input data beyond what is allowed by the first principal component. Towards this end, we consider the one dimensional projection which can best reconstruct the residuals

$$r_i = x_i - (p^T x_i) p_i,$$

that is we replace problem (10.1) by

$$(10.6) \quad \min_{w \in \mathbb{S}^{D-1}, w \perp p} \frac{1}{n} \sum_{i=1}^n \|r_i - (w^T r_i) w\|^2.$$

Note that for all  $i = 1, \dots, n$ ,

$$\|r_i - (w^T r_i) w\|^2 = \|r_i\|^2 - (w^T r_i)^2 = \|r_i\|^2 - (w^T x_i)^2$$

since  $w \perp p$ . Then, following the reasoning from (10.1) to (10.2), problem (10.6) can equivalently be written as

$$(10.7) \quad \max_{w \in \mathbb{S}^{D-1}, w \perp p} \frac{1}{n} \sum_{i=1}^n (w^T x_i)^2 = w^T C_n w.$$

Again, we have to minimize the Rayleigh quotient of the covariance matrix, however, when compared to (10.2), we see that there is the new constraint  $w \perp p$ . Indeed, it can be proven that the solution of problem (10.7) is given by the second eigenvector of  $C_n$ , and the corresponding eigenvalue. The proof of this latter fact follows the same line of the one for the first principal component. Clearly, the above reasoning can be generalized to consider more than two components. The computation of the principal components reduces to the problem of finding the eigenvalues and eigenvectors of  $C_n$ . The complexity of this problem is roughly  $O(kD^2)$  (note that the complexity of forming  $C_n$  is  $O(nD^2)$ ).

The principal components can be stacked as columns of a  $k$  by  $D$  matrix  $M$ , and in fact, because of the orthogonality constraint, the matrix  $M$  is orthogonal,  $MM^T = I$ . The dimensionality reduction induced by PCA is hence linear.

### 10.5. Singular Value Decomposition

We recall the notion of singular valued decomposition of a matrix which allows in some situations to improve the computations of the principal components, while suggesting a possible way to generalize the algorithm to consider non linear dimensionality reduction.

Consider the data matrix  $X_n$ , its singular value decomposition is given by

$$X_n = U\Sigma P^T.$$

where  $U$  is a  $n$  by  $d$  orthogonal matrix,  $P$  is a  $D$  by  $d$  orthogonal matrix,  $\Sigma$  is a diagonal matrix such that  $\Sigma_{i,i} = \sqrt{\lambda_i}$ ,  $i = 1, \dots, d$  and  $d \leq \min\{n, D\}$ . The columns of  $U$  and the columns of  $V$  are called respectively the left and right singular vectors and the diagonal entries of  $\Sigma$  the singular values. The singular value decomposition can be equivalently described by the following equations, for  $j = 1, \dots, d$ ,

$$(10.8) \quad \begin{aligned} C_n p_j &= \lambda_j p_j, & \frac{1}{n} K_n u_j &= \lambda_j u_j, \\ X_n p_j &= \sqrt{\lambda_j} u_j, & \frac{1}{n} X_n^T u_j &= \sqrt{\lambda_j} p_j, \end{aligned}$$

where  $C_n = \frac{1}{n} X_n^T X_n$  and  $\frac{1}{n} K_n = \frac{1}{n} X_n X_n^T$ .

If  $n \ll p$  the above equations can be used to speed up the computation of the principal components. Indeed we can consider the following procedure:

- form the matrix  $K_n$ , which is  $O(Dn^2)$ ,
- find the first  $k$  eigenvectors of  $K_n$ , which is  $O(kn^2)$ ,
- find the principal components using (10.8), i.e.

$$(10.9) \quad p_j = \frac{1}{\sqrt{\lambda_j}} X_n^T u_j = \frac{1}{\sqrt{\lambda_j}} \sum_{i=1}^n x_i u_j^i, \quad j = 1, \dots, d$$

where  $u = (u^1, \dots, u^n)$ , which is again  $O(knd)$  if we consider  $k$  principal components.

### 10.6. Kernel PCA

The latter reasoning suggests how to generalize the intuition behind PCA beyond linear dimensionality reduction by using kernels (or feature maps). Indeed, from equation (10.9) we can see that the projection of a point  $x$  on a principal component  $p$  can be written as

$$(10.10) \quad (M(x))^j = x^T p_j = \frac{1}{\sqrt{\lambda_j}} x^T X_n^T u_j = \frac{1}{\sqrt{\lambda_j}} \sum_{i=1}^n x^T x_i u_j^i,$$

for  $j = 1, \dots, d$ .

What if we were to map the data using a possibly non linear feature map  $\Phi : X \rightarrow F$ , before performing PCA? If the feature map is finite dimensional, e.g.  $F = \mathbb{R}^p$  we could simply replace  $x \mapsto \Phi(x)$  and follow exactly reasoning in the previous sections. Note that in particular that equation (10.10) becomes

$$(10.11) \quad (M(x))^j = \Phi(x)^T p_j = \frac{1}{\sqrt{\lambda_j}} \sum_{i=1}^n \Phi(x)^T \Phi(x_i) u_j^i,$$

for  $j = 1, \dots, d$ . More generally one could consider a positive definite kernel  $K : X \times X \rightarrow \mathbb{R}$ , in which case (10.10) becomes

$$(10.12) \quad (M(x))^j = \frac{1}{\sqrt{\lambda_j}} \sum_{i=1}^n K(x, x_i) u_j^i,$$

for  $j = 1, \dots, d$ . Note that in this latter case, while it is not clear how to form  $C_n$ , we can still form and diagonalize  $K_n$ , which is in fact the kernel matrix.





## Variable Selection

In many practical situations beyond predictions it is important to obtain interpretable results. Interpretability is often determined by detecting which factors have determined our prediction. We look at this question from the perspective of variable selection.

Consider a linear model

$$(11.1) \quad f_w(x) = w^T x = \sum_{i=1}^v w^i x^i.$$

Here we can think of the components  $x^j$  of an input of specific measurements: pixel values in the case of images, dictionary words counting in the case of texts, etc. Given a training set the goal of variable selection is to detect which variables are important for prediction. The key assumption is that the best possible prediction rule is sparse, that is only few of the coefficients in (11.1) are different from zero.

### 11.1. Subset Selection

A brute force approach would be to consider all the training sets obtained considering all the possible subset of variables. More precisely we could begin by considering only the training set where we retain only the first variable of each input points. Then the one where we retain only the second, and so on and so forth. Next, we could pass to consider training set with pairs of variables, then triplet etc. For each training set one would solve the learning problem and eventually end selecting the variables for which the corresponding training set achieve the best performance.

The approach described has an exponential complexity and becomes unfeasible already for relatively small  $D$ . If we consider the square loss, it can be shown that the corresponding problem could be written as

$$(11.2) \quad \min_{w \in \mathbb{R}^D} \frac{1}{n} \sum_{i=1}^n \ell(y_i, f_w(x_i)) + \lambda \|w\|_0,$$

where  $\|w\|_0 = |\{j \mid w^j \neq 0\}|$  is called the  $\ell_0$  norm and counts the number of non zero components in  $w$ . In the following we focus on the least squares loss and consider different approaches to find approximate solution to the above problem, namely greedy methods and convex relaxation.

### 11.2. Greedy Methods: (Orthogonal) Matching Pursuit

Greedy approaches are often considered to find approximate solution to problem (11.2) This class of approaches to variable selection generally encompasses the following steps:

- (1) initialize the residual, the coefficient vector, and the index set,
- (2) find the variable most correlated with the residual,
- (3) update the index set to include the index of such variable,
- (4) update/compute coefficient vector,
- (5) update residual.

The simplest such procedure is called forward stage-wise regression in statistics and matching pursuit (MP) in signal processing. To describe the procedure we need some notation. Let  $X_n$  be the  $n$  by  $D$  data matrix and  $X^j \in \mathbb{R}^n$ ,  $j = 1, \dots, D$  be the columns of  $X_n$ . Let  $Y_n \in \mathbb{R}^n$  be the output vector. Let  $r, w, I$  denote the residual, the coefficient vector, an index set, respectively.

The MP algorithm starts by initializing the residual  $r \in \mathbb{R}^n$ , the coefficient vector  $w \in \mathbb{R}^D$ , and the index set  $I \subseteq \{1, \dots, D\}$ ,

$$r_0 = Y_n, \quad w_0 = 0, \quad I_0 = \emptyset.$$

The following procedure is then iterated for  $i = 1, \dots, T - 1$ . The variable most correlated with the residual is given by

$$k = \arg \max_{j=1, \dots, D} a_j, \quad a_j = \frac{(r_{i-1}^T X^j)^2}{\|X^j\|^2},$$

where we note that

$$v^j = \frac{r_{i-1}^T X^j}{\|X^j\|^2} = \arg \min_{v \in \mathbb{R}} \|r_{i-1} - X^j v\|^2, \quad \|r_{i-1} - X^j v^j\|^2 = \|r_{i-1}\|^2 - a_j$$

The selection rule has then two interpretations. We select the variable, such that the projection on the output on the corresponding column is larger, or, equivalently, we select the variable such that the corresponding column best explains the the output vector in a least squares sense.

Then, index set is updated as  $I_i = I_{i-1} \cup \{k\}$ , and the coefficients vector is given by

$$(11.3) \quad w_i = w_{i-1} + w_k, \quad w_k k = v_k e_k$$

where  $e_k$  is the element of the canonical basis in  $\mathbb{R}^D$  with  $k$ -th component different from zero. Finally, the residual is updated

$$r_i = r_{i-1} - X w^k.$$

A variant of the above procedure, called Orthogonal Matching Pursuit, is also often considered. The corresponding iteration is analogous to that of MP but the coefficient computation (11.3) is replaced by

$$w_i = \arg \min_{w \in \mathbb{R}^D} \|Y_n - X_n M_I w\|^2,$$

where the  $D$  by  $D$  matrix  $M_I$  is such that  $(M_I w)^j = w^j$  if  $j \in I$  and  $(M_I w)^j = 0$  otherwise. Moreover, the residual update is replaced by

$$r_i = Y_n - X_n w_i.$$

### 11.3. Convex Relaxation: LASSO & Elastic Net

Another popular approach to find an approximate solution to problem (11.2) is based on a convex relaxation. Namely, the  $\ell_0$  norm is replaced by the  $\ell_1$  norm,

$$\|w\|_1 = \sum_{j=1}^D |w^j|,$$

so that, in the case of least squares, problem (11.2) is replaced by

$$(11.4) \quad \min_{w \in \mathbb{R}^D} \frac{1}{n} \sum_{i=1}^n (y_i - f_w(x_i))^2 + \lambda \|w\|_1.$$

The above problem is called LASSO in statistics and Basis Pursuit in signal processing. In The objective function defining the corresponding minimization problem is convex but not differentiable. Tools from non-smooth convex optimization are needed to find a solution. A simple yet powerful procedure to compute a solution is based on the so called iterative soft thresholding algorithm (ISTA). The latter is an iterative procedure where, at each iteration, a non linear soft thresholding operator is applied to a gradient step. More precisely, ISTA is defined by the following iteration

$$w_0 = 0, \quad w_i = S_{\lambda\gamma}(w_{i-1} - \frac{2\gamma}{n} X_n^T (Y_n - X_n w_{i-1})), \quad i = 1, \dots, T_{\max}$$

which should be run until a convergence criterion is met, e.g.  $\|w_i - w_{i-1}\| \leq \epsilon$ , for some precision  $\epsilon$ , or a prescribed maximum number of iteration  $T_{\max}$  is reached. To ensure convergence we should choose the step-size  $\gamma = \frac{n}{2\|X_n^T X_n\|}$ . Note that the argument of the soft thresholding operator corresponds to a step of gradient descent. Indeed,

$$\frac{2}{n} X_n^T (Y_n - X_n w_{i-1})$$

The soft thresholding operator acts component wise on a vector  $w$ , so that

$$S_\alpha(u) = |u| - \alpha|_+ \frac{u}{|u|}.$$

The above expression shows that the coefficients of the solution of problem (11.2) as computed by ISTA can be exactly zero, This can be contrasted with Tikhonov regularization where this is hardly the case.

Indeed, it is possible to see that, on the one hand, while Tikhonov allows to compute a stable solution, in general its solution is not sparse. On the other hand the solution of LASSO, might not be stable. The elastic net algorithm, defined as

$$(11.5) \quad \min_{w \in \mathbb{R}^D} \frac{1}{n} \sum_{i=1}^n (y_i - f_w(x_i))^2 + \lambda(\alpha \|w\|_1 + (1 - \alpha) \|w\|_2^2), \quad \alpha \in [0, 1],$$

can be seen as hybrid algorithm which interpolated between Tikhonov and LASSO. The ISTA procedure can be adapted to solve the elastic net problem, where the gradient descent step incorporates also the derivative of the  $\ell^2$  penalty term. The resulting algorithm is

$$w_0 = 0, \quad w_i = S_{\lambda\alpha\gamma}((1 - \lambda\gamma(1 - \alpha))w_{i-1} - \frac{2\gamma}{n} X_n^T (Y_n - X_n w_{i-1})), \quad i = 1, \dots, T_{\max}$$

To ensure convergence we should choose the step-size  $\gamma = \frac{n}{2(\|X_n^T X_n\| + \lambda(1 - \alpha))}$



## A Glimpse Beyond The Fence

We next try to give brief overview of 1) topics in machine learning that we have not touched upon, 2) some of the current and future challenges in machine learning.

### 12.1. Different Kinds of Data

Different machine learning approaches arise to deal with different kinds of input and output. Recall that, the input/output pairs are assumed to belong to an input space  $X$  and an output space  $Y$ , respectively. We call  $Z = X \times Y$  the data space. We list a few examples of input and output spaces.

- Euclidean/Vector Spaces. Perhaps the simplest example, covering many practical situations is  $X = \mathbb{R}^d$ ,  $d \in \mathbb{N}$ .
- Probability distributions. We could set  $X = \{x \in \mathbb{R}_+^d : \sum_{j=1}^d x^j = 1, d \in \mathbb{N}\}$ , and view elements of the space as probability distributions on a finite set  $\Omega$  of dimension  $d$ . More generally given any probability space  $\Omega$  we can view  $X$  as the space of probability distribution on  $\Omega$ .
- Strings/Words. Given an alphabet  $\Sigma$  of symbols (letters) one could consider  $X = \Sigma^p$ ,  $p \in \mathbb{N}$ , the (finite) space of strings (words) of  $p$  letters.
- Graphs. We can view  $X$  as collection of graphs, i.e.  $X = \{\}$ .

Clearly more exotic examples can be constructed considering compositions of the above examples, for example  $X = \mathbb{R}^d \times \Sigma^p$ ,  $d, p \in \mathbb{N}$  etc.

Next we discuss different choices of the output space and see how they often corresponds to problems with different names.

- Regression,  $Y = \mathbb{R}$ .
- Binary classification,  $Y = \{-1, 1\}$ . Where we note that here we could have taken  $Y = \{0, 1\}$ —as well as any other pair of distinct numeric values.
- Multivariate regression,  $Y = \mathbb{R}^T$ ,  $T \in \mathbb{N}$ , each output is a vector.
- Functional regression,  $Y$  is a Hilbert space, for example each output is a function.
- Multi-category classification,  $Y = \{1, 2, \dots, T\}$ ,  $T \in \mathbb{N}$ , the output is one of  $T$  categories.
- Multilabel,  $Y = 2^{\{1, 2, \dots, T\}}$ ,  $T \in \mathbb{N}$ , each output is any subset of  $T$  categories.

An interesting case is that of so called multitask learning. Here  $Z = (X_1, Y_1) \times (X_2, Y_2) \times \dots \times (X_T, Y_T)$  and the training set is  $S = (S_1, S_2, \dots, S_T)$ . Here we can view each data space/training set as corresponding to different yet related tasks. In full generality, input/output spaces and data cardinality can be different.

### 12.2. Data and Sampling Models

The standard data model we consider is a training set as an i.i.d. sample from a distribution  $p$  on the data space  $Z$ .

- Semisupervised, the more general situation where unlabelled data  $S_u$  are available together with the labelled data  $S$ .
- Transductive, related to the above setting, unlabelled data  $S_u$  are available together with the labelled data and the goal is to predict the label of the unlabeled data set  $S_u$ .
- Online/Dynamic Learning, the data are not i.i.d. The samples can be dependent, the samples can come from varying distribution or both.

### 12.3. Learning Approaches

- Online/Incremental
- Randomized
- distributed

- Online/Dynamic Learning, the data are not i.i.d. The samples can be dependent, the samples can come from varying distribution or both.
- Active,
- Reinforcement Learning.

#### 12.4. Some Current and Future Challenges in Machine Learning

Challenges

$$1 \leftarrow \text{Data Size} \rightarrow \infty + \infty$$

**12.4.1. Big Data?** Recent times have seen the development of technologies for gathering data-set of unprecedented size and complexity both in natural science and technology. On the one hand this has opened novel opportunities (e.g. online teaching), on the other had it has posed new challenges. In particular, the necessity has emerged to develop learning techniques capable to leverage predefined *budgets* and requisites in terms of

- Computations,
- Communications,
- Privacy.

**12.4.2. Or Small Data?** One of the most evident difference between biological and artificial intelligence is the astounding ability of humans to generalize from limited supervised data. Indeed, while impressive, current artificial intelligent systems based on supervised learning required huge amounts of humanly annotated data.

- Unsupervised learning of data representation
- Learning under weak supervision.
- Learning and exploiting structure among learning tasks.

## Mathematical Tools

These notes present a brief summary of some of the basic definitions from calculus that we will need in this class. Throughout these notes, we assume that we are working with the base field  $\mathbb{R}$ .

### A.1. Structures on Vector Spaces

A **vector space**  $V$  is a set with a linear structure. This means we can add elements of the vector space or multiply elements by scalars (real numbers) to obtain another element. A familiar example of a vector space is  $\mathbb{R}^n$ . Given  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  in  $\mathbb{R}^n$ , we can form a new vector  $x + y = (x_1 + y_1, \dots, x_n + y_n) \in \mathbb{R}^n$ . Similarly, given  $r \in \mathbb{R}$ , we can form  $rx = (rx_1, \dots, rx_n) \in \mathbb{R}^n$ .

Every vector space has a basis. A subset  $B = \{v_1, \dots, v_n\}$  of  $V$  is called a **basis** if every vector  $v \in V$  can be expressed uniquely as a linear combination  $v = c_1v_1 + \dots + c_nv_n$  for some constants  $c_1, \dots, c_n \in \mathbb{R}$ . The cardinality (number of elements) of  $B$  is called the **dimension** of  $V$ . This notion of dimension is well defined because while there is no canonical way to choose a basis, all bases of  $V$  have the same cardinality. For example, the standard basis on  $\mathbb{R}^n$  is  $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ . This shows that  $\mathbb{R}^n$  is an  $n$ -dimensional vector space, in accordance with the notation. In this section we will be working with finite dimensional vector spaces only.

We note that any two finite dimensional vector spaces over  $\mathbb{R}$  are isomorphic, since a bijection between the bases can be extended linearly to be an isomorphism between the two vector spaces. Hence, up to isomorphism, for every  $n \in \mathbb{N}$  there is only one  $n$ -dimensional vector space, which is  $\mathbb{R}^n$ . However, vector spaces can also have extra structures that distinguish them from each other, as we shall explore now.

A **distance (metric)** on  $V$  is a function  $d: V \times V \rightarrow \mathbb{R}$  satisfying:

- (positivity)  $d(v, w) \geq 0$  for all  $v, w \in V$ , and  $d(v, w) = 0$  if and only if  $v = w$ .
- (symmetry)  $d(v, w) = d(w, v)$  for all  $v, w \in V$ .
- (triangle inequality)  $d(v, w) \leq d(v, x) + d(x, w)$  for all  $v, w, x \in V$ .

The standard distance function on  $\mathbb{R}^n$  is given by  $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$ . Note that the notion of metric does not require a linear structure, or any other structure, on  $V$ ; a metric can be defined on any set.

A similar concept that requires a linear structure on  $V$  is **norm**, which measures the “length” of vectors in  $V$ . Formally, a norm is a function  $\|\cdot\|: V \rightarrow \mathbb{R}$  that satisfies the following three properties:

- (positivity)  $\|v\| \geq 0$  for all  $v \in V$ , and  $\|v\| = 0$  if and only if  $v = 0$ .
- (homogeneity)  $\|rv\| = |r|\|v\|$  for all  $r \in \mathbb{R}$  and  $v \in V$ .
- (subadditivity)  $\|v + w\| \leq \|v\| + \|w\|$  for all  $v, w \in V$ .

For example, the standard norm on  $\mathbb{R}^n$  is  $\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$ , which is also called the  $\ell_2$ -norm. Also of interest is the  $\ell_1$ -norm  $\|x\|_1 = |x_1| + \dots + |x_n|$ , which we will study later in this class in relation to sparsity-based algorithms. We can also generalize these examples to any  $p \geq 1$  to obtain the  $\ell_p$ -norm, but we will not do that here.

Given a normed vector space  $(V, \|\cdot\|)$ , we can define the **distance (metric) function** on  $V$  to be  $d(v, w) = \|v - w\|$ . For example, the  $\ell_2$ -norm on  $\mathbb{R}^n$  gives the standard distance function

$$d(x, y) = \|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2},$$

while the  $\ell_1$ -norm on  $\mathbb{R}^n$  gives the Manhattan/taxicab distance,

$$d(x, y) = \|x - y\|_1 = |x_1 - y_1| + \dots + |x_n - y_n|.$$

As a side remark, we note that all norms on a finite dimensional vector space  $V$  are **equivalent**. This means that for any two norms  $\mu$  and  $\nu$  on  $V$ , there exist positive constants  $C_1$  and  $C_2$  such that for all  $v \in V$ ,  $C_1\mu(v) \leq \nu(v) \leq C_2\mu(v)$ . In particular, continuity or convergence with respect to one norm

implies continuity or convergence with respect to any other norms in a finite dimensional vector space. For example, on  $\mathbb{R}^n$  we have the inequality  $\|x\|_1/\sqrt{n} \leq \|x\|_2 \leq \|x\|_1$ .

Another structure that we can introduce to a vector space is the inner product. An **inner product** on  $V$  is a function  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$  that satisfies the following properties:

- (symmetry)  $\langle v, w \rangle = \langle w, v \rangle$  for all  $v, w \in V$ .
- (linearity)  $\langle r_1v_1 + r_2v_2, w \rangle = r_1\langle v_1, w \rangle + r_2\langle v_2, w \rangle$  for all  $r_1, r_2 \in \mathbb{R}$  and  $v_1, v_2, w \in V$ .
- (positive-definiteness)  $\langle v, v \rangle \geq 0$  for all  $v \in V$ , and  $\langle v, v \rangle = 0$  if and only if  $v = 0$ .

For example, the standard inner product on  $\mathbb{R}^n$  is  $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n$ , which is also known as the *dot product*, written  $x \cdot y$ .

Given an inner product space  $(V, \langle \cdot, \cdot \rangle)$ , we can define the norm of  $v \in V$  to be  $\|v\| = \sqrt{\langle v, v \rangle}$ . It is easy to check that this definition satisfies the axioms for a norm listed above. On the other hand, not every norm arises from an inner product. The necessary and sufficient condition that has to be satisfied for a norm to be induced by an inner product is the **parallelogram law**:

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

If the parallelogram law is satisfied, then the inner product can be defined by **polarization identity**:

$$\langle v, w \rangle = \frac{1}{4}(\|v + w\|^2 - \|v - w\|^2).$$

For example, you can check that the  $\ell_2$ -norm on  $\mathbb{R}^n$  is induced by the standard inner product, while the  $\ell_1$ -norm is not induced by an inner product since it does not satisfy the parallelogram law.

A very important result involving inner product is the following **Cauchy-Schwarz inequality**:

$$\langle v, w \rangle \leq \|v\|\|w\| \text{ for all } v, w \in V.$$

Inner product also allows us to talk about orthogonality. Two vectors  $v$  and  $w$  in  $V$  are said to be **orthogonal** if  $\langle v, w \rangle = 0$ . In particular, an **orthonormal basis** is a basis  $v_1, \dots, v_n$  that is orthogonal ( $\langle v_i, v_j \rangle = 0$  for  $i \neq j$ ) and normalized ( $\langle v_i, v_i \rangle = 1$ ). Given an orthonormal basis  $v_1, \dots, v_n$ , the decomposition of  $v \in V$  in terms of this basis has the special form

$$v = \sum_{i=1}^n \langle v, v_i \rangle v_i.$$

For example, the standard basis vectors  $e_1, \dots, e_n$  form an orthonormal basis of  $\mathbb{R}^n$ . In general, a basis  $v_1, \dots, v_n$  can be orthonormalized using the Gram-Schmidt process.

Given a subspace  $W$  of an inner product space  $V$ , we can define the **orthogonal complement** of  $W$  to be the set of all vectors in  $V$  that are orthogonal to  $W$ ,

$$W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

If  $V$  is finite dimensional, then we have the **orthogonal decomposition**  $V = W \oplus W^\perp$ . This means every vector  $v \in V$  can be decomposed uniquely into  $v = w + w'$ , where  $w \in W$  and  $w' \in W^\perp$ . The vector  $w$  is called the **projection** of  $v$  on  $W$ , and represents the unique vector in  $W$  that is closest to  $v$ .

## A.2. Matrices

In addition to talking about vector spaces, we can also talk about operators on those spaces. A **linear operator** is a function  $L: V \rightarrow W$  between two vector spaces that preserves the linear structure. In finite dimension, every linear operator can be represented by a matrix by choosing a basis in both the domain and the range, i.e. by working in coordinates. For this reason we focus the first part of our discussion on matrices.

If  $V$  is  $n$ -dimensional and  $W$  is  $m$ -dimensional, then a linear map  $L: V \rightarrow W$  is represented by an  $m \times n$  matrix  $A$  whose columns are the values of  $L$  applied to the basis of  $V$ . The **rank** of  $A$  is the dimension of the image of  $A$ , and the **nullity** of  $A$  is the dimension of the kernel of  $A$ . The **rank-nullity theorem** states that  $\text{rank}(A) + \text{nullity}(A) = m$ , the dimension of the domain of  $A$ . Also note that the transpose of  $A$  is an  $n \times m$  matrix  $A^\top$  satisfying

$$\langle Av, w \rangle_{\mathbb{R}^m} = (Av)^\top w = v^\top A^\top w = \langle v, A^\top w \rangle_{\mathbb{R}^n}$$

for all  $v \in \mathbb{R}^n$  and  $w \in \mathbb{R}^m$ .

Let  $A$  be an  $n \times n$  matrix with real entries. Recall that an **eigenvalue**  $\lambda \in \mathbb{R}$  of  $A$  is a solution to the equation  $Av = \lambda v$  for some nonzero vector  $v \in \mathbb{R}^n$ , and  $v$  is the **eigenvector** of  $A$  corresponding to  $\lambda$ . If  $A$  is symmetric, i.e.  $A^\top = A$ , then the eigenvalues of  $A$  are real. Moreover, in this case the **spectral theorem**



tells us that there is an orthonormal basis of  $\mathbb{R}^n$  consisting of the eigenvectors of  $A$ . Let  $v_1, \dots, v_n$  be this orthonormal basis of eigenvectors, and let  $\lambda_1, \dots, \lambda_n$  be the corresponding eigenvalues. Then we can write

$$A = \sum_{i=1}^n \lambda_i v_i v_i^\top,$$

which is called the **eigendecomposition** of  $A$ . We can also write this as

$$A = V \Lambda V^\top,$$

where  $V$  is the  $n \times n$  matrix with columns  $v_i$ , and  $\Lambda$  is the  $n \times n$  diagonal matrix with entries  $\lambda_i$ . The orthonormality of  $v_1, \dots, v_n$  makes  $V$  an orthogonal matrix, i.e.  $V^{-1} = V^\top$ .

A symmetric  $n \times n$  matrix  $A$  is **positive definite** if  $v^\top A v > 0$  for all nonzero vectors  $v \in \mathbb{R}^n$ .  $A$  is **positive semidefinite** if the inequality is not strict (i.e.  $\geq 0$ ). A positive definite (resp. positive semidefinite) matrix  $A$  has positive (resp. nonnegative) eigenvalues.

Another method for decomposing a matrix is the **singular value decomposition** (SVD). Given an  $m \times n$  real matrix  $A$ , the SVD of  $A$  is the factorization

$$A = U \Sigma V^\top,$$

where  $U$  is an  $m \times m$  orthogonal matrix ( $U^\top U = I$ ),  $\Sigma$  is an  $m \times n$  diagonal matrix, and  $V$  is an  $n \times n$  orthogonal matrix ( $V^\top V = I$ ). The columns  $u_1, \dots, u_m$  of  $U$  form an orthonormal basis of  $\mathbb{R}^m$ , and the columns  $v_1, \dots, v_n$  of  $V$  form an orthonormal basis of  $\mathbb{R}^n$ . The diagonal elements  $\sigma_1, \dots, \sigma_{\min\{m,n\}}$  in  $\Sigma$  are nonnegative and called the **singular values** of  $A$ . This factorization corresponds to the decomposition

$$A = \sum_{i=1}^{\min\{m,n\}} \sigma_i u_i v_i^\top.$$

This decomposition shows the relations between  $\sigma_i$ ,  $u_i$ , and  $v_i$  more clearly: for  $1 \leq i \leq \min\{m, n\}$ ,

$$\begin{aligned} A v_i &= \sigma_i u_i & A A^\top u_i &= \sigma_i^2 u_i \\ A^\top u_i &= \sigma_i v_i & A^\top A v_i &= \sigma_i^2 v_i \end{aligned}$$

This means the  $u_i$ 's are eigenvectors of  $A A^\top$  with corresponding eigenvalues  $\sigma_i^2$ , and the  $v_i$ 's are eigenvectors of  $A^\top A$ , also with corresponding eigenvalues  $\sigma_i^2$ .

Given an  $m \times n$  matrix  $A$ , we can define the **spectral norm** of  $A$  to be largest singular value of  $A$ ,

$$\|A\|_{\text{spec}} = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A A^\top)} = \sqrt{\lambda_{\max}(A^\top A)}.$$

Another common norm on  $A$  is the **Frobenius norm**,

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2} = \sqrt{\text{trace}(A A^\top)} = \sqrt{\text{trace}(A^\top A)} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2}.$$

However, since the space of all matrices can be identified with  $\mathbb{R}^{m \times n}$ , the discussion in Section A.1 still holds and all norms on  $A$  are equivalent.